Proof Let $G' = \{a_i' \mid i \in I\}$, and let $A = \{a_i \mid i \in I\}$ be a set with the same number of elements as G'. Let $G = F[A]$. Then by Theorem 39.12 there exists a homomorphism ψ mapping G into G' such that $\psi(a_i) = a_i'$. Clearly the image of G under ψ is all of G'. ◆

Another Look at Free Abelian Groups

It is important that we do not confuse the notion of a free group with the notion of a free abelian group. A free group on more than one generator is not abelian. In the preceding section, we defined a free abelian group as an abelian group that has a basis, that is, a generating set satisfying properties described in Theorem 38.1. There is another approach, via free groups, to free abelian groups. We now describe this approach.

Let $F[A]$ be the free group on the generating set A. We shall write F in place of $F[A]$ for the moment. Note that F is not abelian if A contains more than one element. Let C be the commutator subgroup of F. Then F/C is an abelian group, and it is not hard to show that F/C is free abelian with basis $\{aC \mid a \in A\}$. If aC is renamed a, we can view F/C as a free abelian group with basis A. This indicates how a free abelian group having a given set as basis can be constructed. Every free abelian group can be constructed in this fashion, up to isomorphism. That is, if G is free abelian with basis X, form the free group $F[X]$, form the factor group of $F[X]$ modulo its commutator subgroup, and we have a group isomorphic to G.

Theorems 39.7, 39.9, and 39.10 hold for free abelian groups as well as for free groups. In fact, the abelian version of Theorem 39.10 was proved for the finite rank case in Theorem 38.11. In contrast to Example 39.11 for free groups, it is true that for a free abelian group the rank of a subgroup is at most the rank of the entire group. Theorem 38.11 also showed this for the finite rank case.

EXERCISES 39

Computations

1. Find the reduced form and the inverse of the reduced form of each of the following words.

   a. $a^2b^{-1}b^3a^3c^{-1}c^4b^{-2}$ **b.** $a^2a^{-3}b^3a^4c^4c^2a^{-1}$

2. Compute the products given in parts (a) and (b) of Exercise 1 in the case that $\{a, b, c\}$ is a set of generators forming a basis for a free abelian group. Find the inverse of these products.

3. How many different homomorphisms are there of a free group of rank 2 into

   a. $\mathbb{Z}_4$? **b.** $\mathbb{Z}_6$? **c.** S_3?

4. How many different homomorphisms are there of a free group of rank 2 onto

   a. $\mathbb{Z}_4$? **b.** $\mathbb{Z}_6$? **c.** S_3?

5. How many different homomorphisms are there of a free abelian group of rank 2 into

   a. $\mathbb{Z}_4$? **b.** $\mathbb{Z}_6$? **c.** S_3?

6. How many different homomorphisms are there of a free abelian group of rank 2 onto

   a. $\mathbb{Z}_4$? **b.** $\mathbb{Z}_6$? **c.** S_3?

Concepts

In Exercises 7 and 8, correct the definition of the italicized term without reference to the text, if correction is needed. so that it is in a form acceptable for publication.

7. A *reduced word* is one in which there are no appearances in juxtaposition of two syllables having the same letter and also no appearances of a syllable with exponent 0.
8. The *rank of a free group* is the number of elements in a set of generators for the group.
9. Take one of the instances in this section in which the phrase "It would seem obvious that" was used and discuss your reaction in that instance.
10. Mark each of the following true or false.

    _______ **a.** Every proper subgroup of a free group is a free group.
    _______ **b.** Every proper subgroup of every free abelian group is a free group.
    _______ **c.** A homomorphic image of a free group is a free group.
    _______ **d.** Every free abelian group has a basis.
    _______ **e.** The free abelian groups of finite rank are precisely the finitely generated abelian groups.
    _______ **f.** No free group is free.
    _______ **g.** No free abelian group is free.
    _______ **h.** No free abelian group of rank >1 is free.
    _______ **i.** Any two free groups are isomorphic.
    _______ **j.** Any two free abelian groups of the same rank are isomorphic.

Theory

11. Let G be a finitely generated abelian group with identity 0. A finite set $\{b_1, \cdots, b_n\}$, where $b_i \in G$, is a **basis for** G if $\{b_1, \cdots, b_n\}$ generates G and $\sum_{i=1}^{n} m_i b_i = 0$ if and only if each $m_i b_i = 0$, where $m_i \in \mathbb{Z}$.
    a. Show that $\{2, 3\}$ is not a basis for $\mathbb{Z}_4$. Find a basis for $\mathbb{Z}_4$.
    b. Show that both $\{1\}$ and $\{2, 3\}$ are bases for $\mathbb{Z}_6$. (This shows that for a finitely generated abelian group G with torsion, the number of elements in a basis may vary; that is, it need not be an *invariant* of the group G.)
    c. Is a basis for a free abelian group as we defined it in Section 38 a basis in the sense in which it is used in this exercise?
    d. Show that every finite abelian group has a basis $\{b_1, \cdots, b_n\}$, where the order of b_i divides the order of b_{i+1}.

In present-day expositions of algebra, a frequently used technique (particularly by the disciples of N. Bourbaki) for introducing a new algebraic entity is the following:

1. Describe algebraic properties that this algebraic entity is to possess.
2. Prove that any two algebraic entities with these properties are isomorphic, that is, that these properties characterize the entity.
3. Show that at least one such entity exists.

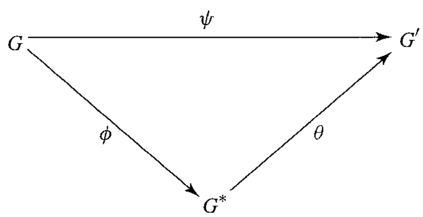
The next three exercises illustrate this technique for three algebraic entities, each of which we have met before. So that we do not give away their identities, we use fictitious names for them in the first two exercises. The last part of these first two exercises asks us to give the usual name for the entity.

12. Let G be any group. An abelian group G^* is a **blip group of** G if there exists a fixed homomorphism ϕ of G onto G^* such that each homomorphism ψ of G into an abelian group G' can be factored as $\psi = \theta\phi$, where θ is a homomorphism of G^* into G' (see Fig. 39.14).

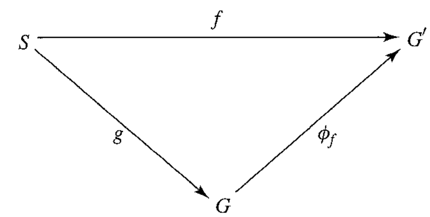
   a. Show that any two blip groups of G are isomorphic. [*Hint:* Let ${G_1}^*$ and ${G_2}^*$ be two blip groups of G. Then each of the fixed homomorphisms $\phi_1 : G \to {G_1}^*$ and $\phi_2 : G \to {G_2}^*$ can be factored via the other blip group according to the definition of a blip group; that is, $\phi_1 = \theta_1\phi_2$ and $\phi_2 = \theta_2\phi_1$. Show that θ_1 is an isomorphism of ${G_2}^*$ onto ${G_1}^*$ by showing that both $\theta_1\theta_2$ and $\theta_2\theta_1$ are identity maps.]

   b. Show for every group G that a blip group G^* of G exists.

   c. What concept that we have introduced before corresponds to this idea of a blip group of G?



39.14 Figure



39.15 Figure

13. Let S be any set. A group G together with a fixed function $g : S \to G$ constitutes a **blop group on** S if for each group G' and map $f : S \to G'$ there exists a *unique* homomorphism ϕ_f of G into G' such that $f = \phi_f g$ (see Fig. 39.15).

   a. Let S be a fixed set. Show that if both G_1, together with $g_1 : S \to G_1$, and G_2, together with $g_2 : S \to G_2$, are blop groups on S, then G_1 and G_2 are isomorphic. [*Hint:* Show that g_1 and g_2 are one-to-one maps and that g_1S and g_2S generate G_1 and G_2, respectively. Then proceed in a way analogous to that given by the hint for Exercise 12.]

   b. Let S be a set. Show that a blop group on S exists. You may use any theorems of the text.

   c. What concept that we have introduced before corresponds to this idea of a blop group on S?

14. Characterize a free abelian group by properties in a fashion similar to that used in Exercise 13.

SECTION 40 Group Presentations

Definition

Following most of the literature on group presentations, in this section we let 1 be the identity of a group. The idea of a *group presentation* is to form a group by giving a set of generators for the group and certain equations or relations that we want the generators to satisfy. We want the group to be as free as it possibly can be on the generators, subject to these relations.

40.1 Example Suppose G has generators x and y and is *free except for the relation* $xy = yx$, which we may express as $xyx^{-1}y^{-1} = 1$. Note that the condition $xy = yx$ is exactly what is needed to make G abelian, even though $xyx^{-1}y^{-1}$ is just one of the many possible commutators of $F[\{x, y\}]$. Thus G is free abelian on two generators and is isomorphic to $F[\{x, y\}]$ modulo its commutator subgroup. This commutator subgroup of $F[\{x, y\}]$ is the smallest normal subgroup containing $xyx^{-1}y^{-1}$, since any normal subgroup