be either 18 or 36. If the order is 18, the normalizer is then of index 2 and therefore is normal in G. If the order is 36, then $H \cap K$ is normal in G. ▲

37.15 Example Every group of order $255 = (3)(5)(17)$ is abelian (hence cyclic by the Fundamental Theorem 11.12 and not simple, since 255 is not a prime). By Theorem 36.11 such a group G has only one subgroup H of order 17. Then G/H has order 15 and is abelian by Example 37.10. By Theorem 15.20, we see that the commutator subgroup C of G is contained in H. Thus as a subgroup of H, C has either order 1 or 17. Theorem 36.11 also shows that G has either 1 or 85 subgroups of order 3 and either 1 or 51 subgroups of order 5. However, 85 subgroups of order 3 would require 170 elements of order 3, and 51 subgroups of order 5 would require 204 elements of order 5 in G; both together would then require 375 elements in G, which is impossible. Hence there is a subgroup K having either order 3 or order 5 and normal in G. Then G/K has either order (5)(17) or order (3)(17), and in either case Theorem 37.7 shows that G/K is abelian. Thus $C \leq K$ and has order either 3, 5, or 1. Since $C \leq H$ showed that C has order 17 or 1, we conclude that C has order 1. Hence $C = \{e\}$, and $G/C \simeq G$ is abelian. The Fundamental Theorem 11.12 then shows that G is cyclic. ▲

EXERCISES 37

Computations

1. Let D_4 be the group of symmetries of the square in Example 8.10.
   a. Find the decomposition of D_4 into conjugate classes.
   b. Write the class equation for D_4.
2. By arguments similar to those used in the examples of this section, convince yourself that every nontrivial group of order not a prime and less than 60 contains a nontrivial proper normal subgroup and hence is not simple. You need not write out the details. (The hardest cases were discussed in the examples.)

Concepts

3. Mark each of the following true or false.
   ______ **a.** Every group of order 159 is cyclic.
   ______ **b.** Every group of order 102 has a nontrivial proper normal subgroup.
   ______ **c.** Every solvable group is of prime-power order.
   ______ **d.** Every group of prime-power order is solvable.
   ______ **e.** It would become quite tedious to show that no group of nonprime order between 60 and 168 is simple by the methods illustrated in the text.
   ______ **f.** No group of order 21 is simple.
   ______ **g.** Every group of 125 elements has at least 5 elements that commute with every element in the group.
   ______ **h.** Every group of order 42 has a normal subgroup of order 7.
   ______ **i.** Every group of order 42 has a normal subgroup of order 8.
   ______ **j.** The only simple groups are the groups $\mathbb{Z}_p$ and A_n where p is a prime and $n \neq 4$.

Theory

4. Prove that every group of order (5)(7)(47) is abelian and cyclic.

5. Prove that no group of order 96 is simple.

6. Prove that no group of order 160 is simple.

7. Show that every group of order 30 contains a subgroup of order 15. [*Hint:* Use the last sentence in Example 37.12, and go to the factor group.]

8. This exercise determines the conjugate classes of S_n for every integer $n \geq 1$.

a. Show that if $\sigma = (a_1, a_2, \cdots, a_m)$ is a cycle in S_n and τ is any element of S_n then $\tau\sigma\tau^{-1} = (\tau a_1, \tau a_2, \cdots, \tau a_m)$.

b. Argue from (a) that any two cycles in S_n of the same length are conjugate.

c. Argue from (a) and (b) that a product of s disjoint cycles in S_n of lengths r_i for $i = 1, 2, \cdots, s$ is conjugate to every other product of s disjoint cycles of lengths r_i in S_n.

d. Show that the number of conjugate classes in S_n is $p(n)$, where $p(n)$ is the number of ways, neglecting the order of the summands, that n can be expressed as a sum of positive integers. The number $p(n)$ is the **number of partitions of** n.

e. Compute $p(n)$ for $n = 1, 2, 3, 4, 5, 6, 7$.

9. Find the conjugate classes and the class equation for S_4. [*Hint:* Use Exercise 8.]

10. Find the class equation for S_5 and S_6. [*Hint:* Use Exercise 8.]

11. Show that the number of conjugate classes in S_n is also the number of different abelian groups (up to isomorphism) of order p^n, where p is a prime number. [*Hint:* Use Exercise 8.]

12. Show that if $n > 2$, the center of S_n is the subgroup consisting of the identity permutation only. [*Hint:* Use Exercise 8.]

SECTION 38 FREE ABELIAN GROUPS

In this section we introduce the concept of free abelian groups and prove some results concerning them. The section concludes with a demonstration of the Fundamental Theorem of finitely generated abelian groups (Theorem 11.12).

Free Abelian Groups

We should review the notions of a generating set for a group G and a finitely generated group, as given in Section 7. In this section we shall deal exclusively with abelian groups and use additive notations as follows:

$$0 \text{ for the identity, } + \text{ for the operation,}$$

$$\left.\begin{aligned} na &= \underbrace{a + a + \cdots + a}_{n \text{ summands}} \\ -na &= \underbrace{(-a) + (-a) + \cdots + (-a)}_{n \text{ summands}} \end{aligned}\right\} \text{for } n \in \mathbb{Z}^+ \text{ and } a \in G.$$

$$0a = 0 \text{ for the first 0 in } \mathbb{Z} \text{ and the second in } G.$$

We shall continue to use the symbol $\times$ for direct product of groups rather than change to direct sum notation.