**34.10 Example** Consider $K = 6\mathbb{Z} < H = 2\mathbb{Z} < G = \mathbb{Z}$. Then $G/H = \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2$. Now $G/K = \mathbb{Z}/6\mathbb{Z}$ has elements

$$6\mathbb{Z}, \quad 1+6\mathbb{Z}, \quad 2+6\mathbb{Z}, \quad 3+6\mathbb{Z}, \quad 4+6\mathbb{Z}, \quad \text{and} \quad 5+6\mathbb{Z}.$$

Of these six cosets, $6\mathbb{Z}$, $2+6\mathbb{Z}$, and $4+6\mathbb{Z}$ lie in $2\mathbb{Z}/6\mathbb{Z}$. Thus $(\mathbb{Z}/6\mathbb{Z})/(2\mathbb{Z}/6\mathbb{Z})$ has two elements and is isomorphic to $\mathbb{Z}_2$ also. Alternatively, we see that $\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}_6$, and $2\mathbb{Z}/6\mathbb{Z}$ corresponds *under this isomorphism* to the cyclic subgroup $\langle 2 \rangle$ of $\mathbb{Z}_6$. Thus $(\mathbb{Z}/6\mathbb{Z})/(2\mathbb{Z}/6\mathbb{Z}) \simeq \mathbb{Z}_6/\langle 2 \rangle \simeq \mathbb{Z}_2 \simeq \mathbb{Z}/2\mathbb{Z}$. ▲

## ■ EXERCISES 34

### Computations

In using the three isomorphism theorems, it is often necessary to know the actual correspondence given by the isomorphism and not just the fact that the groups are isomorphic. The first six exercises give us training for this.

**1.** Let $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_3$ be the homomorphism such that $\phi(1) = 2$.

- **a.** Find the kernel $K$ of $\phi$.
- **b.** List the cosets in $\mathbb{Z}_{12}/K$, showing the elements in each coset.
- **c.** Give the correspondence between $\mathbb{Z}_{12}/K$ and $\mathbb{Z}_3$ given by the map $\mu$ described in Theorem 34.2.

**2.** Let $\phi : \mathbb{Z}_{18} \to \mathbb{Z}_{12}$ be the homomorphism where $\phi(1) = 10$.

- **a.** Find the kernel $K$ of $\phi$.
- **b.** List the cosets in $\mathbb{Z}_{18}/K$, showing the elements in each coset.
- **c.** Find the group $\phi[\mathbb{Z}_{18}]$.
- **d.** Give the correspondence between $\mathbb{Z}_{18}/K$ and $\phi[\mathbb{Z}_{18}]$ given by the map $\mu$ described in Theorem 34.2.

**3.** In the group $\mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and $N = \langle 6 \rangle$.

- **a.** List the elements in $HN$ (which we might write $H + N$ for these additive groups) and in $H \cap N$.
- **b.** List the cosets in $HN/N$, showing the elements in each coset.
- **c.** List the cosets in $H/(H \cap N)$, showing the elements in each coset.
- **d.** Give the correspondence between $HN/N$ and $H/(H \cap N)$ described in the proof of Theorem 34.5.

**4.** Repeat Exercise 3 for the group $\mathbb{Z}_{36}$ with $H = \langle 6 \rangle$ and $N = \langle 9 \rangle$.

**5.** In the group $G = \mathbb{Z}_{24}$, let $H = \langle 4 \rangle$ and $K = \langle 8 \rangle$.

- **a.** List the cosets in $G/H$, showing the elements in each coset.
- **b.** List the cosets in $G/K$, showing the elements in each coset.
- **c.** List the cosets in $H/K$, showing the elements in each coset.
- **d.** List the cosets in $(G/K)/(H/K)$, showing the elements in each coset.
- **e.** Give the correspondence between $G/H$ and $(G/K)/(H/K)$ described in the proof of Theorem 34.7.

**6.** Repeat Exercise 5 for the group $G = \mathbb{Z}_{36}$ with $H = \langle 9 \rangle$ and $K = \langle 18 \rangle$.

### Theory

**7.** Show directly from the definition of a normal subgroup that if $H$ and $N$ are subgroups of a group $G$, and $N$ is normal in $G$, then $H \cap N$ is normal in $H$.

**8.** Let $H$, $K$, and $L$ be normal subgroups of $G$ with $H < K < L$. Let $A = G/H$, $B = K/H$, and $C = L/H$.

**a.** Show that $B$ and $C$ are normal subgroups of $A$, and $B < C$.

**b.** To what factor group of $G$ is $(A/B)/(C/B)$ isomorphic?

**9.** Let $K$ and $L$ be normal subgroups of $G$ with $K \vee L = G$, and $K \cap L = \{e\}$. Show that $G/K \simeq L$ and $G/L \simeq K$.

## SECTION 35 SERIES OF GROUPS

### Subnormal and Normal Series

This section is concerned with the notion of a *series* of a group $G$, which gives insight into the structure of $G$. The results hold for both abelian and nonabelian groups. They are not too important for finitely generated abelian groups because of our strong Theorem 11.12. Many of our illustrations will be taken from abelian groups, however, for ease of computation.

**35.1 Definition** A **subnormal** (or **subinvariant**) **series of a group** $G$ is a finite sequence $H_0, H_1, \cdots, H_n$ of subgroups of $G$ such that $H_i < H_{i+1}$ and $H_i$ is a normal subgroup of $H_{i+1}$ with $H_0 = \{e\}$ and $H_n = G$. A **normal** (or **invariant**) **series of** $G$ is a finite sequence $H_0, H_1, \cdots, H_n$ of normal subgroups of $G$ such that $H_i < H_{i+1}$, $H_0 = \{e\}$, and $H_n = G$. ■

Note that for abelian groups the notions of subnormal and normal series coincide, since every subgroup is normal. A normal series is always subnormal, but the converse need not be true. We defined a subnormal series before a normal series, since the concept of a subnormal series is more important for our work.

**35.2 Example** Two examples of normal series of $\mathbb{Z}$ under addition are

$$\{0\} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}$$

and

$$\{0\} < 9\mathbb{Z} < \mathbb{Z}.$$

▲

**35.3 Example** Consider the group $D_4$ of symmetries of the square in Example 8.10. The series

$$\{\rho_0\} < \{\rho_0, \mu_1\} < \{\rho_0, \rho_2, \mu_1, \mu_2\} < D_4$$

is a subnormal series, as we could check using Table 8.12. It is not a normal series since $\{\rho_0, \mu_1\}$ is not normal in $D_4$. ▲

**35.4 Definition** A subnormal (normal) series $\{K_j\}$ is a **refinement of a subnormal (normal) series** $\{H_i\}$ of a group $G$ if $\{H_i\} \subseteq \{K_j\}$, that is, if each $H_i$ is one of the $K_j$. ■

**35.5 Example** The series

$$\{0\} < 72\mathbb{Z} < 24\mathbb{Z} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}$$