26.19 Example Example 26.11 shows that $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$, so we can form the factor ring $\mathbb{Z}/n\mathbb{Z}$. Example 18.11 shows that $\phi : \mathbb{Z} \rightarrow \mathbb{Z}_n$ where $\phi(m)$ is the remainder of m modulo n is a homomorphism, and we see that $\text{Ker}(\phi) = n\mathbb{Z}$. Theorem 26.17 then shows that the map $\mu : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}_n$ where $\mu(m + n\mathbb{Z})$ is the remainder of m modulo n is well defined and is an isomorphism. ▲

In summary, every ring homomorphism with domain R gives rise to a factor ring R/N, and every factor ring R/N gives rise to a homomorphism mapping R into R/N. An *ideal* in ring theory is analogous to a *normal subgroup* in the group theory. Both are the type of substructure needed to form a factor structure.

We should now add an addendum to Theorem 26.3 on properties of homomorphisms. Let $\phi : R \rightarrow R'$ be a homomorphism, and let N be an ideal of R. Then $\phi[N]$ is an ideal of $\phi[R]$, although it need not be an ideal of R'. Also, if N' is an ideal of either $\phi[R]$ or of R', then $\phi^{-1}[N']$ is an ideal of R. We leave the proof of this to Exercise 22.

EXERCISES 26

Computations

1. Describe all ring homomorphisms of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{Z} \times \mathbb{Z}$. [*Hint:* Note that if ϕ is such a homomorphism, then $\phi((1, 0)) = \phi((1, 0))\phi((1, 0))$ and $\phi((0, 1)) = \phi((0, 1))\phi((0, 1))$. Consider also $\phi((1, 0)(0, 1))$.]
2. Find all positive integers n such that $\mathbb{Z}_n$ contains a subring isomorphic to $\mathbb{Z}_2$.
3. Find all ideals N of $\mathbb{Z}_{12}$. In each case compute $\mathbb{Z}_{12}/N$; that is, find a known ring to which the quotient ring is isomorphic.
4. Give addition and multiplication tables for $2\mathbb{Z}/8\mathbb{Z}$. Are $2\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}_4$ isomorphic rings?

Concepts

In Exercises 5 through 7, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

5. An *isomorphism of a ring* R with a ring R' is a homomorphism $\phi : R \rightarrow R'$ such that $\text{Ker}(\phi) = \{0\}$.
6. An *ideal* N of a ring R is an additive subgroup of $\langle R, +\rangle$ such that for all $r \in R$ and all $n \in N$, we have $rn \in N$ and $nr \in N$.
7. The *kernel of a homomorphism* ϕ mapping a ring R into a ring R' is $\{\phi(r) = 0' \mid r \in R\}$.
8. Let F be the ring of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ and having derivatives of all orders. Differentiation gives a map $\delta : F \rightarrow F$ where $\delta(f(x)) = f'(x)$. Is δ a homomorphism? Why? Give the connection between this exercise and Example 26.12.
9. Give an example of a ring homomorphism $\phi : R \rightarrow R'$ where R has unity 1 and $\phi(1) \neq 0'$, but $\phi(1)$ is not unity for R'.
10. Mark each of the following true or false.

    ______ **a.** The concept of a ring homomorphism is closely connected with the idea of a factor ring.
    ______ **b.** A ring homomorphism $\phi : R \rightarrow R'$ carries ideals of R into ideals of R'.
    ______ **c.** A ring homomorphism is one to one if and only if the kernel is $\{0\}$.
    ______ **d.** $\mathbb{Q}$ is an ideal in $\mathbb{R}$.

_______ **e.** Every ideal in a ring is a subring of the ring.
_______ **f.** Every subring of every ring is an ideal of the ring.
_______ **g.** Every quotient ring of every commutative ring is again a commutative ring.
_______ **h.** The rings $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}_4$ are isomorphic.
_______ **i.** An ideal N in a ring R with unity 1 is all of R if and only if $1 \in N$.
_______ **j.** The concept of an ideal is to the concept of a ring as the concept of a normal subgroup is to the concept of a group.

11. Let R be a ring. Observe that $\{0\}$ and R are both ideals of R. Are the factor rings R/R and $R/\{0\}$ of real interest? Why?

12. Give an example to show that a factor ring of an integral domain may be a field.

13. Give an example to show that a factor ring of an integral domain may have divisors of 0.

14. Give an example to show that a factor ring of a ring with divisors of 0 may be an integral domain.

15. Find a subring of the ring $\mathbb{Z} \times \mathbb{Z}$ that is not an ideal of $\mathbb{Z} \times \mathbb{Z}$.

16. A student is asked to prove that a quotient ring of a ring R modulo an ideal N is commutative if and only if $(rs - sr) \in N$ for all $r, s \in R$. The student starts out:
Assume R/N is commutative. Then $rs = sr$ for all $r, s \in R/N$.

a. Why does the instructor reading this expect nonsense from there on?
b. What should the student have written?
c. Prove the assertion. (Note the "if and only if.")

Theory

17. Let $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ and let R' consist of all 2×2 matrices of the form $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{Z}$. Show that R is a subring of $\mathbb{R}$ and that R' is a subring of $M_2(\mathbb{Z})$. Then show that $\phi : R \to R'$, where $\phi(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ is an isomorphism.

18. Show that each homomorphism from a field to a ring is either one to one or maps everything onto 0.

19. Show that if R, R', and R'' are rings, and if $\phi : R \to R'$ and $\psi : R' \to R''$ are homomorphisms, then the composite function $\psi\phi : R \to R''$ is a homomorphism. (Use Exercise 49 of Section 13.)

20. Let R be a commutative ring with unity of prime characteristic p. Show that the map $\phi_p : R \to R$ given by $\phi_p(a) = a^p$ is a homomorphism (the **Frobenius homomorphism**).

21. Let R and R' be rings and let $\phi : R \to R'$ be a ring homomorphism such that $\phi[R] \neq \{0'\}$. Show that if R has unity 1 and R' has no 0 divisors, then $\phi(1)$ is unity for R'.

22. Let $\phi : R \to R'$ be a ring homomorphism and let N be an ideal of R.

a. Show that $\phi[N]$ is an ideal of $\phi[R]$.
b. Give an example to show that $\phi[N]$ need not be an ideal of R'.
c. Let N' be an ideal either of $\phi[R]$ or of R'. Show that $\phi^{-1}[N']$ is an ideal of R.

23. Let F be a field, and let S be any subset of $F \times F \times \cdots \times F$ for n factors. Show that the set N_S of all $f(x_1, \cdots, x_n) \in F[x_1, \cdots, x_n]$ that have every element $(a_1, \cdots, a_n)$ of S as a zero (see Exercise 28 of Section 22) is an ideal in $F[x_1, \cdots, x_n]$. This is of importance in algebraic geometry.

24. Show that a factor ring of a field is either the trivial (zero) ring of one element or is isomorphic to the field.

25. Show that if R is a ring with unity and N is an ideal of R such that $N \neq R$, then R/N is a ring with unity.

26. Let R be a commutative ring and let $a \in R$. Show that $I_a = \{x \in R \mid ax = 0\}$ is an ideal of R.

27. Show that an intersection of ideals of a ring R is again an ideal of R.

28. Let R and R' be rings and let N and N' be ideals of R and R', respectively. Let ϕ be a homomorphism of R into R'. Show that ϕ induces a natural homomorphism $\phi_* : R/N \to R'/N'$ if $\phi[N] \subseteq N'$. (Use Exercise 39 of Section 14.)

29. Let ϕ be a homomorphism of a ring R with unity onto a nonzero ring R'. Let u be a unit in R. Show that $\phi(u)$ is a unit in R'.

30. An element a of a ring R is **nilpotent** if $a^n = 0$ for some $n \in \mathbb{Z}^+$. Show that the collection of all nilpotent elements in a commutative ring R is an ideal, the **nilradical of** R.

31. Referring to the definition given in Exercise 30, find the nilradical of the ring $\mathbb{Z}_{12}$ and observe that it is one of the ideals of $\mathbb{Z}_{12}$ found in Exercise 3. What is the nilradical of $\mathbb{Z}$? of $\mathbb{Z}_{32}$?

32. Referring to Exercise 30, show that if N is the nilradical of a commutative ring R, then R/N has as nilradical the trivial ideal $\{0 + N\}$.

33. Let R be a commutative ring and N an ideal of R. Referring to Exercise 30, show that if every element of N is nilpotent and the nilradical of R/N is R/N, then the nilradical of R is R.

34. Let R be a commutative ring and N an ideal of R. Show that the set $\sqrt{N}$ of all $a \in R$, such that $a^n \in N$ for some $n \in \mathbb{Z}^+$, is an ideal of R, the **radical of** N.

35. Referring to Exercise 34, show by examples that for proper ideals N of a commutative ring R,

a. $\sqrt{N}$ need not equal N **b.** $\sqrt{N}$ may equal N.

36. What is the relationship of the ideal $\sqrt{N}$ of Exercise 34 to the nilradical of R/N (see Exercise 30)? Word your answer carefully.

37. Show that $\phi : \mathbb{C} \to M_2(\mathbb{R})$ given by

$$\phi(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

for $a, b \in \mathbb{R}$ gives an isomorphism of $\mathbb{C}$ with the subring $\phi[\mathbb{C}]$ of $M_2(\mathbb{R})$.

38. Let R be a ring with unity and let $\mathrm{End}(\langle R, +\rangle)$ be the ring of endomorphisms of $\langle R, +\rangle$ as described in Section 24. Let $a \in R$, and let $\lambda_a : R \to R$ be given by

$$\lambda_a(x) = ax$$

for $x \in R$.

a. Show that λ_a is an endomorphism of $\langle R, +\rangle$.

b. Show that $R' = \{\lambda_a \mid a \in R\}$ is a subring of $\mathrm{End}(\langle R, +\rangle)$.

c. Prove the analogue of Cayley's theorem for R by showing that R' of (b) is isomorphic to R.

SECTION 27 Prime and Maximal Ideals

Exercises 12 through 14 of the preceding section asked us to provide examples of factor rings R/N where R and R/N have very different structural properties. We start with some examples of this situation, and in the process, provide solutions to those exercises.

27.1 Example As was shown in Corollary 19.12, the ring $\mathbb{Z}_p$, which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, is a field for p a prime. *Thus a factor ring of an integral domain may be a field.* ▲