and

$$\psi(x + y) = \psi(x) + \psi(y)$$

follow easily from the definition of ψ on F and from the fact that ψ is the identity on D.

If $\psi(a /_F b) = \psi(c /_F d)$, we have

$$\psi(a) /_L \psi(b) = \psi(c) /_L \psi(d)$$

so

$$\psi(a)\psi(d) = \psi(b)\psi(c).$$

Since ψ is the identity on D, we then deduce that $ad = bc$, so $a /_F b = c /_F d$. Thus ψ is one to one.

By definition, $\psi(a) = a$ for $a \in D$. ◆

21.8 Corollary Every field L containing an integral domain D contains a field of quotients of D.

Proof In the proof of Theorem 21.6 every element of the subfield $\psi[F]$ of L is a quotient in L of elements of D. ◆

21.9 Corollary Any two fields of quotients of an integral domain D are isomorphic.

Proof Suppose in Theorem 21.6 that L is a field of quotients of D, so that every element x of L can be expressed in the form $a /_L b$ for $a, b \in D$. Then L is the field $\psi[F]$ of the proof of Theorem 21.6 and is thus isomorphic to F. ◆

EXERCISES 21

Computations

1. Describe the field F of quotients of the integral subdomain

$$D = \{n + mi \mid n, m \in \mathbb{Z}\}$$

of $\mathbb{C}$. "Describe" means give the elements of $\mathbb{C}$ that make up the field of quotients of D in $\mathbb{C}$. (The elements of D are the **Gaussian integers**.)

2. Describe (in the sense of Exercise 1) the field F of quotients of the integral subdomain $D = \{n + m\sqrt{2} \mid n, m \in \mathbb{Z}\}$ of $\mathbb{R}$.

Concepts

3. Correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

A *field of quotients* of an integral domain D is a field F in which D can be embedded so that every nonzero element of D is a unit in F.

4. Mark each of the following true or false.

______ a. $\mathbb{Q}$ is a field of quotients of $\mathbb{Z}$.

______ b. $\mathbb{R}$ is a field of quotients of $\mathbb{Z}$.

______ c. $\mathbb{R}$ is a field of quotients of $\mathbb{R}$.

______ d. $\mathbb{C}$ is a field of quotients of $\mathbb{R}$.

______ e. If D is a field, then any field of quotients of D is isomorphic to D.

______ f. The fact that D has no divisors of 0 was used strongly several times in the construction of a field F of quotients of the integral domain D.

______ g. Every element of an integral domain D is a unit in a field F of quotients of D.

______ h. Every nonzero element of an integral domain D is a unit in a field F of quotients of D.

______ i. A field of quotients F' of a subdomain D' of an integral domain D can be regarded as a subfield of some field of quotients of D.

______ j. Every field of quotients of $\mathbb{Z}$ is isomorphic to $\mathbb{Q}$.

5. Show by an example that a field F' of quotients of a proper subdomain D' of an integral domain D may also be a field of quotients for D.

Theory

6. Prove Part 2 of Step 3. You may assume any preceding part of Step 3.
7. Prove Part 3 of Step 3. You may assume any preceding part of Step 3.
8. Prove Part 4 of Step 3. You may assume any preceding part of Step 3.
9. Prove Part 5 of Step 3. You may assume any preceding part of Step 3.
10. Prove Part 6 of Step 3. You may assume any preceding part of Step 3.
11. Prove Part 7 of Step 3. You may assume any preceding part of Step 3.
12. Let R be a nonzero commutative ring, and let T be a nonempty subset of R closed under multiplication and containing neither 0 nor divisors of 0. Starting with $R \times T$ and otherwise exactly following the construction in this section, we can show that the ring R can be enlarged to a *partial ring of quotients* $Q(R, T)$. Think about this for 15 minutes or so; look back over the construction and see why things still work. In particular, show the following:

    a. $Q(R, T)$ has unity even if R does not.

    b. In $Q(R, T)$, every nonzero element of T is a unit.
13. Prove from Exercise 12 that every nonzero commutative ring containing an element a that is not a divisor of 0 can be enlarged to a commutative ring with unity. Compare with Exercise 30 of Section 19.
14. With reference to Exercise 12, how many elements are there in the ring $Q(\mathbb{Z}_4, \{1, 3\})$?
15. With reference to Exercise 12, describe the ring $Q(\mathbb{Z}, \{2^n \mid n \in \mathbb{Z}^+\})$, by describing a subring of $\mathbb{R}$ to which it is isomorphic.
16. With reference to Exercise 12, describe the ring $Q(3\mathbb{Z}, \{6^n \mid n \in \mathbb{Z}^+\})$ by describing a subring of $\mathbb{R}$ to which it is isomorphic.
17. With reference to Exercise 12, suppose we drop the condition that T have no divisors of zero and just require that nonempty T not containing 0 be closed under multiplication. The attempt to enlarge R to a commutative ring with unity in which every nonzero element of T is a unit must fail if T contains an element a that is a divisor of 0, for a divisor of 0 cannot also be a unit. Try to discover where a construction parallel to that in the text but starting with $R \times T$ first runs into trouble. In particular, for $R = \mathbb{Z}_6$ and $T = \{1, 2, 4\}$, illustrate the first difficulty encountered. [*Hint:* It is in Step 1.]