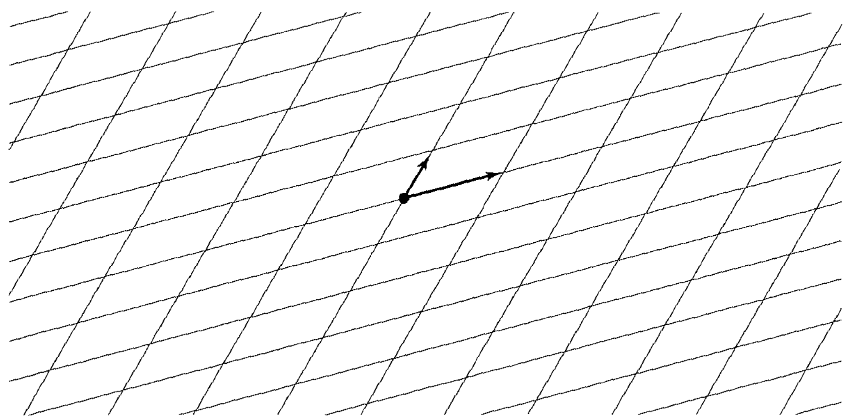


12.7 Figure

generated by the translations indicated by the arrows and a rotation through 180° about any vertex of a parallelogram.

It can be shown that there are 17 different types of wallpaper patterns when they are classified according to the types of rotations, reflections, and nontrivial glide reflections that they admit. We refer you to Gallian [8] for pictures of these 17 possibilities and a chart to help you identify them. The exercises illustrate a few of them. The situation in space is more complicated; it can be shown that there are 230 three-dimensional crystallographic groups. The final exercise we give involves rotations in space.

M. C. Escher (1898–1973) was an artist whose work included plane-filling patterns. The exercises include reproductions of four of his works of this type.

EXERCISES 12

1. This exercise shows that the group of symmetries of a certain type of geometric figure may depend on the dimension of the space in which we consider the figure to lie.

a. Describe all symmetries of a point in the real line $\mathbb{R}$; that is, describe all isometries of $\mathbb{R}$ that leave one point fixed.

b. Describe all symmetries (translations, reflections, etc.) of a point in the plane $\mathbb{R}^2$.

c. Describe all symmetries of a line segment in $\mathbb{R}$.

d. Describe all symmetries of a line segment in $\mathbb{R}^2$.

e. Describe some symmetries of a line segment in $\mathbb{R}^3$.

2. Let P stand for an orientation preserving plane isometry and R for an orientation reversing one. Fill in the table with P or R to denote the orientation preserving or reversing property of a product.

	P	R
P		
R		

3. Fill in the table to give *all* possible types of plane isometries given by a product of two types. For example, a product of two rotations may be a rotation, or it may be another type. Fill in the box corresponding to $\rho\rho$ with both letters. Use your answer to Exercise 2 to eliminate some types. Eliminate the identity from consideration.

	τ	ρ	μ	γ
τ				
ρ				
μ				
γ				

4. Draw a plane figure that has a one-element group as its group of symmetries in $\mathbb{R}^2$.

5. Draw a plane figure that has a two-element group as its group of symmetries in $\mathbb{R}^2$.

6. Draw a plane figure that has a three-element group as its group of symmetries in $\mathbb{R}^2$.

7. Draw a plane figure that has a four-element group isomorphic to $\mathbb{Z}_4$ as its group of symmetries in $\mathbb{R}^2$.

8. Draw a plane figure that has a four-element group isomorphic to the Klein 4-group V as its group of symmetries in $\mathbb{R}^2$.

9. For each of the four types of plane isometries (other than the identity), give the possibilities for the order of an isometry of that type in the group of plane isometries.

10. A plane isometry ϕ has a *fixed point* if there exists a point P in the plane such that $\phi(P) = P$. Which of the four types of plane isometries (other than the identity) can have a fixed point?

11. Referring to Exercise 10, which types of plane isometries, if any, have exactly one fixed point?

12. Referring to Exercise 10, which types of plane isometries, if any, have exactly two fixed points?

13. Referring to Exercise 10, which types of plane isometries, if any, have an infinite number of fixed points?

14. Argue geometrically that a plane isometry that leaves three noncolinear points fixed must be the identity map.

15. Using Exercise 14, show algebraically that if two plane isometries ϕ and ψ agree on three noncolinear points, that is, if $\phi(P_i) = \psi(P_i)$ for noncolinear points P_1, P_2, and P_3, then ϕ and ψ are the same map.

16. Do the rotations, together with the identity map, form a subgroup of the group of plane isometries? Why or why not?

17. Do the translations, together with the identity map, form a subgroup of the group of plane isometries? Why or why not?

18. Do the rotations about one particular point P, together with the identity map, form a subgroup of the group of plane isometries? Why or why not?

19. Does the reflection across one particular line L, together with the identity map, form a subgroup of the group of plane isometries? Why or why not?

20. Do the glide reflections, together with the identity map, form a subgroup of the group of plane isometries? Why or why not?

21. Which of the four types of plane isometries can be elements of a *finite* subgroup of the group of plane isometries?

22. Completing a detail of the proof of Theorem 12.5, let G be a finite group of plane isometries. Show that the rotations in G, together with the identity isometry, form a subgroup H of G, and that either $H = G$ or $|G| = 2|H|$. [*Hint:* Use the same method that we used to show that $|S_n| = 2|A_n|$.]

23. Completing a detail in the proof of Theorem 12.5, let G be a finite group consisting of the identity isometry and rotations about one point P in the plane. Show that G is cyclic, generated by the rotation in G that turns the plane counterclockwise about P through the smallest angle $\theta > 0$. [*Hint:* Follow the idea of the proof that a subgroup of a cyclic group is cyclic.]

Exercises 24 through 30 illustrate the seven different types of friezes when they are classified according to their symmetries. Imagine the figure shown to be continued infinitely to the right and left. The symmetry group of a frieze always contains translations. For each of these exercises answer these questions about the symmetry group of the frieze.

a. Does the group contain a rotation?

b. Does the group contain a reflection across a horizontal line?

c. Does the group contain a reflection across a vertical line?

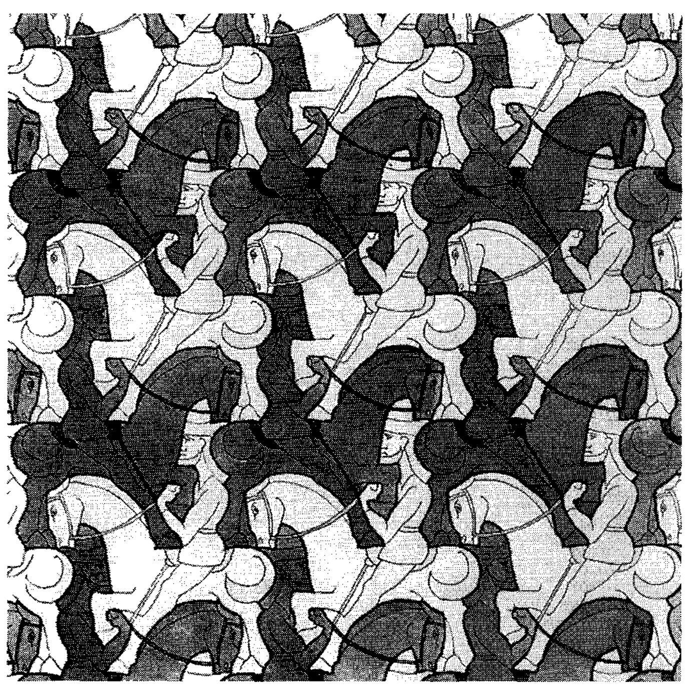
d. Does the group contain a nontrivial glide reflection?

e. To which of the possible groups $\mathbb{Z}$, D_∞, $\mathbb{Z} \times \mathbb{Z}_2$, or $D_\infty \times \mathbb{Z}_2$ do you think the symmetry group of the frieze is isomorphic?

24. F F F F F F F F F F F F F F

25. T T T T T T T T T T

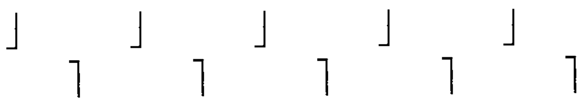
26. E E E E E E E E E E E E



12.8 Figure The Study of Regular Division of the Plane with Horsemen (© 1946 M. C. Escher Foundation–Baarn–Holland. All rights reserved.)

27. Z Z Z Z Z Z Z Z Z Z Z Z Z

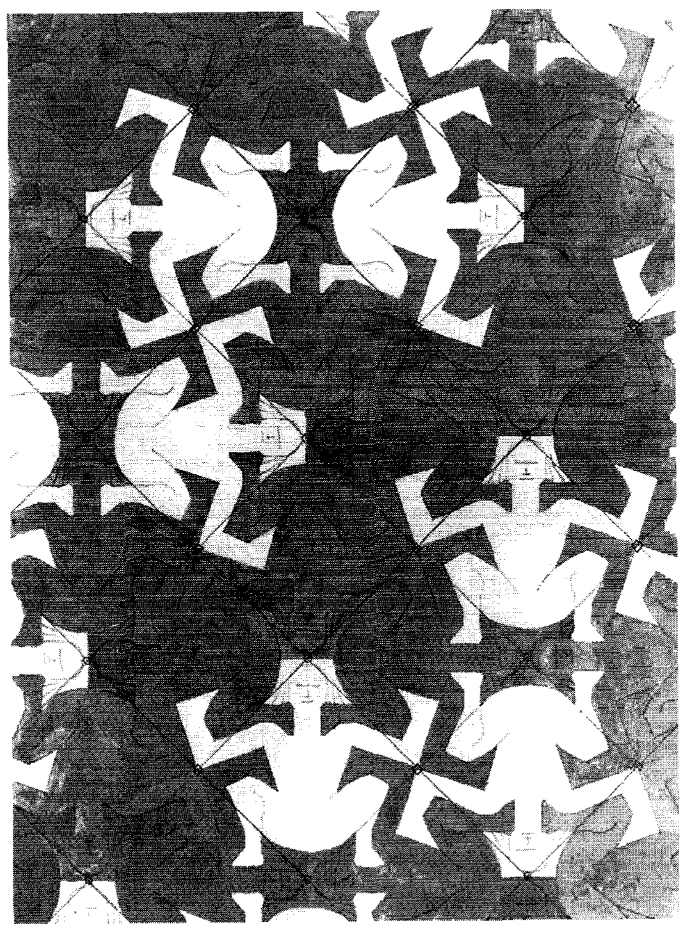
28. H H H H H H H H H H

29. 

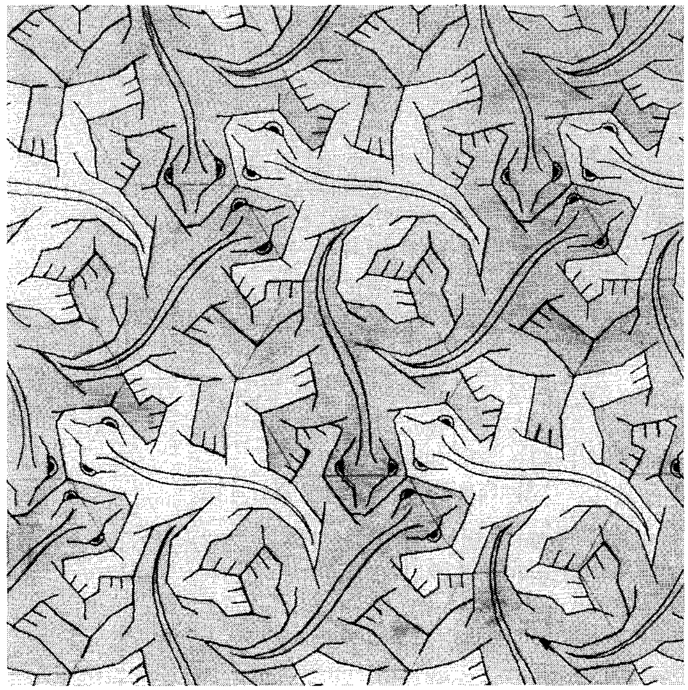
30. 

Exercises 31 through 37 describe a pattern to be used to fill the plane by translation in the two directions given by the specified vectors. Answer these questions in each case.

a. Does the symmetry group contain any rotations? If so, through what possible angles θ where $0 < \theta \leq 180°$?



12.9 Figure The Study of Regular Division of the Plane with Imaginary Human Figures (© 1936 M. C. Escher Foundation–Baarn–Holland. All rights reserved.)



12.10 Figure The Study of Regular Division of the Plane with Reptiles (© 1939 M. C. Escher Foundation–Baarn–Holland. All rights reserved.)

b. Does the symmetry group contain any reflections?

c. Does the symmetry group contain any nontrivial glide reflections?

31. A square with horizontal and vertical edges using translation directions given by vectors $(1, 0)$ and $(0, 1)$.

32. A square as in Exercise 31 using translation directions given by vectors $(1, 1/2)$ and $(0, 1)$.

33. A square as in Exercise 31 with the letter L at its center using translation directions given by vectors $(1, 0)$ and $(0, 1)$.

34. A square as in Exercise 31 with the letter E at its center using translation directions given by vectors $(1, 0)$ and $(0, 1)$.

35. A square as in Exercise 31 with the letter H at its center using translation directions given by vectors $(1, 0)$ and $(0, 1)$.

36. A regular hexagon with a vertex at the top using translation directions given by vectors $(1, 0)$ and $(1, \sqrt{3})$.

37. A regular hexagon with a vertex at the top containing an equilateral triangle with vertex at the top and centroid at the center of the hexagon, using translation directions given by vectors $(1, 0)$ and $(1, \sqrt{3})$.

Exercises 38 through 41 are concerned with art works of M. C. Escher. Neglect the shading in the figures and assume the markings in each human figure, reptile, or horseman are the same, even though they may be invisible due to shading. Answer the same questions (a), (b), and (c) that were asked for Exercises 31 through 36, and also answer this part (d).

d. Assuming horizontal and vertical coordinate axes with equal scales as usual, give vectors in the two nonparallel directions of vectors that generate the translation subgroup. Do not concern yourself with the length of these vectors.