where not all primes p_i need be distinct. Since $(p_1)^{r_1}(p_2)^{r_2}\cdots(p_n)^{r_n}$ is the order of G, then m must be of the form $(p_1)^{s_1}(p_2)^{s_2}\cdots(p_n)^{s_n}$, where $0 \le s_i \le r_i$. By Theorem 6.14, $(p_i)^{r_i-s_i}$ generates a cyclic subgroup of $\mathbb{Z}_{(p_i)^{r_i}}$ of order equal to the quotient of $(p_i)^{r_i}$ by the gcd of $(p_i)^{r_i}$ and $(p_i)^{r_i-s_i}$. But the gcd of $(p_i)^{r_i}$ and $(p_i)^{r_i-s_i}$ is $(p_i)^{r_i-s_i}$. Thus $(p_i)^{r_i-s_i}$ generates a cyclic subgroup $\mathbb{Z}_{(p_i)^{r_i}}$ of order

$$[(p_i)^{r_i}]/[(p_i)^{r_i-s_i}] = (p_i)^{s_i}.$$

Recalling that $\langle a\rangle$ denotes the cyclic subgroup generated by a, we see that

$$\langle (p_1)^{r_1-s_1}\rangle \times \langle (p_2)^{r_2-s_2}\rangle \times \cdots \times \langle (p_n)^{r_n-s_n}\rangle$$

is the required subgroup of order m. ◆

11.17 Theorem If m is a square free integer, that is, m is not divisible by the square of any prime, then every abelian group of order m is cyclic.

Proof Let G be an abelian group of square free order m. Then by Theorem 11.12, G is isomorphic to

$$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \cdots \times \mathbb{Z}_{(p_n)^{r_n}},$$

where $m = (p_1)^{r_1}(p_2)^{r_2}\cdots(p_n)^{r_n}$. Since m is square free, we must have all $r_i = 1$ and all p_i distinct primes. Corollary 11.6 then shows that G is isomorphic to $\mathbb{Z}_{p_1p_2\cdots p_n}$, so G is cyclic. ◆

EXERCISES 11

Computations

1. List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Find the order of each of the elements. Is this group cyclic?
2. Repeat Exercise 1 for the group $\mathbb{Z}_3 \times \mathbb{Z}_4$.

In Exercises 3 through 7, find the order of the given element of the direct product.

3. (2, 6) in $\mathbb{Z}_4 \times \mathbb{Z}_{12}$
4. (2, 3) in $\mathbb{Z}_6 \times \mathbb{Z}_{15}$
5. (8, 10) in $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$
6. (3, 10, 9) in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$
7. (3, 6, 12, 16) in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{20} \times \mathbb{Z}_{24}$
8. What is the largest order among the orders of all the cyclic subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_8$? of $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$?
9. Find all proper nontrivial subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.
10. Find all proper nontrivial subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
11. Find all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$ of order 4.
12. Find all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ that are isomorphic to the Klein 4-group.
13. Disregarding the order of the factors, write direct products of two or more groups of the form $\mathbb{Z}_n$ so that the resulting product is isomorphic to $\mathbb{Z}_{60}$ in as many ways as possible.
14. Fill in the blanks.
    a. The cyclic subgroup of $\mathbb{Z}_{24}$ generated by 18 has order___.
    b. $\mathbb{Z}_3 \times \mathbb{Z}_4$ is of order___.

c. The element (4, 2) of $\mathbb{Z}_{12} \times \mathbb{Z}_8$ has order__.

d. The Klein 4-group is isomorphic to $\mathbb{Z}_{\_\_} \times \mathbb{Z}_{\_\_}$.

e. $\mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}_4$ has__elements of finite order.

15. Find the maximum possible order for some element of $\mathbb{Z}_4 \times \mathbb{Z}_6$.

16. Are the groups $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ isomorphic? Why or why not?

17. Find the maximum possible order for some element of $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$.

18. Are the groups $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$ isomorphic? Why or why not?

19. Find the maximum possible order for some element of $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$.

20. Are the groups $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}$ isomorphic? Why or why not?

In Exercises 21 through 25, proceed as in Example 11.13 to find all abelian groups, up to isomorphism, of the given order.

21. Order 8 **22.** Order 16 **23.** Order 32

24. Order 720 **25.** Order 1089

26. How many abelian groups (up to isomorphism) are there of order 24? of order 25? of order (24)(25)?

27. Following the idea suggested in Exercise 26, let m and n be relatively prime positive integers. Show that if there are (up to isomorphism) r abelian groups of order m and s of order n, then there are (up to isomorphism) rs abelian groups of order mn.

28. Use Exercise 27 to determine the number of abelian groups (up to isomorphism) of order $(10)^5$.

29. **a.** Let p be a prime number. Fill in the second row of the table to give the number of abelian groups of order p^n, up to isomorphism.

n	2	3	4	5	6	7	8
number of groups							

b. Let p, q, and r be distinct prime numbers. Use the table you created to find the number of abelian groups, up to isomorphism, of the given order.

i. $p^3q^4r^7$ **ii.** $(qr)^7$ **iii.** $q^5r^4q^3$

30. Indicate schematically a Cayley digraph for $\mathbb{Z}_m \times \mathbb{Z}_n$ for the generating set $S = \{(1, 0), (0, 1)\}$.

31. Consider Cayley digraphs with two arc types, a solid one with an arrow and a dashed one with no arrow, and consisting of two regular n-gons, for $n \geq 3$, with solid arc sides, one inside the other, with dashed arcs joining the vertices of the outer n-gon to the inner one. Figure 7.9(b) shows such a Cayley digraph with $n = 3$, and Figure 7.11(b) shows one with $n = 4$. The arrows on the outer n-gon may have the same (clockwise or counterclockwise) direction as those on the inner n-gon, or they may have the opposite direction. Let G be a group with such a Cayley digraph.

a. Under what circumstances will G be abelian?

b. If G is abelian, to what familiar group is it isomorphic?

c. If G is abelian, under what circumstances is it cyclic?

d. If G is not abelian, to what group we have discussed is it isomorphic?

Concepts

32. Mark each of the following true or false.

_____ **a.** If G_1 and G_2 are any groups, then $G_1 \times G_2$ is always isomorphic to $G_2 \times G_1$.

_____ **b.** Computation in an external direct product of groups is easy if you know how to compute in each component group.

_____ **c.** Groups of finite order must be used to form an external direct product.

_____ **d.** A group of prime order could not be the internal direct product of two proper nontrivial subgroups.

_____ **e.** $\mathbb{Z}_2 \times \mathbb{Z}_4$ is isomorphic to $\mathbb{Z}_8$.

_____ **f.** $\mathbb{Z}_2 \times \mathbb{Z}_4$ is isomorphic to S_8.

_____ **g.** $\mathbb{Z}_3 \times \mathbb{Z}_8$ is isomorphic to S_4.

_____ **h.** Every element in $\mathbb{Z}_4 \times \mathbb{Z}_8$ has order 8.

_____ **i.** The order of $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$ is 60.

_____ **j.** $\mathbb{Z}_m \times \mathbb{Z}_n$ has mn elements whether m and n are relatively prime or not.

33. Give an example illustrating that not every nontrivial abelian group is the internal direct product of two proper nontrivial subgroups.

34. **a.** How many subgroups of $\mathbb{Z}_5 \times \mathbb{Z}_6$ are isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_6$?

b. How many subgroups of $\mathbb{Z} \times \mathbb{Z}$ are isomorphic to $\mathbb{Z} \times \mathbb{Z}$?

35. Give an example of a nontrivial group that is not of prime order and is not the internal direct product of two nontrivial subgroups.

36. Mark each of the following true or false.

_____ **a.** Every abelian group of prime order is cyclic.

_____ **b.** Every abelian group of prime power order is cyclic.

_____ **c.** $\mathbb{Z}_8$ is generated by $\{4, 6\}$.

_____ **d.** $\mathbb{Z}_8$ is generated by $\{4, 5, 6\}$.

_____ **e.** All finite abelian groups are classified up to isomorphism by Theorem 11.12.

_____ **f.** Any two finitely generated abelian groups with the same Betti number are isomorphic.

_____ **g.** Every abelian group of order divisible by 5 contains a cyclic subgroup of order 5.

_____ **h.** Every abelian group of order divisible by 4 contains a cyclic subgroup of order 4.

_____ **i.** Every abelian group of order divisible by 6 contains a cyclic subgroup of order 6.

_____ **j.** Every finite abelian group has a Betti number of 0.

37. Let p and q be distinct prime numbers. How does the number (up to isomorphism) of abelian groups of order p^r compare with the number (up to isomorphism) of abelian groups of order q^r?

38. Let G be an abelian group of order 72.

a. Can you say how many subgroups of order 8 G has? Why, or why not?

b. Can you say how many subgroups of order 4 G has? Why, or why not?

39. Let G be an abelian group. Show that the elements of finite order in G form a subgroup. This subgroup is called the **torsion subgroup** of G.

Exercises 40 through 43 deal with the concept of the torsion subgroup just defined.

40. Find the order of the torsion subgroup of $\mathbb{Z}_4 \times \mathbb{Z} \times \mathbb{Z}_3$; of $\mathbb{Z}_{12} \times \mathbb{Z} \times \mathbb{Z}_{12}$.

41. Find the torsion subgroup of the multiplicative group $\mathbb{R}^*$ of nonzero real numbers.

42. Find the torsion subgroup T of the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers.

43. An abelian group is **torsion free** if e is the only element of finite order. Use Theorem 11.12 to show that every finitely generated abelian group is the internal direct product of its torsion subgroup and of a torsion-free subgroup. (Note that $\{e\}$ may be the torsion subgroup, and is also torsion free.)

44. The part of the decomposition of G in Theorem 11.12 corresponding to the subgroups of prime-power order can also be written in the form $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}$, where m_i divides m_{i+1} for $i = 1, 2, \cdots, r-1$. The numbers m_i can be shown to be unique, and are the **torsion coefficients** of G.

a. Find the torsion coefficients of $\mathbb{Z}_4 \times \mathbb{Z}_9$.

b. Find the torsion coefficients of $\mathbb{Z}_6 \times \mathbb{Z}_{12} \times \mathbb{Z}_{20}$.

c. Describe an algorithm to find the torsion coefficients of a direct product of cyclic groups.

Proof Synopsis

45. Give a two-sentence synopsis of the proof of Theorem 11.5.

Theory

46. Prove that a direct product of abelian groups is abelian.

47. Let G be an abelian group. Let H be the subset of G consisting of the identity e together with all elements of G of order 2. Show that H is a subgroup of G.

48. Following up the idea of Exercise 47 determine whether H will always be a subgroup for every abelian group G if H consists of the identity e together with all elements of G of order 3; of order 4. For what positive integers n will H always be a subgroup for every abelian group G, if H consists of the identity e together with all elements of G of order n? Compare with Exercise 48 of Section 5.

49. Find a counterexample of Exercise 47 with the hypothesis that G is abelian omitted.

Let H and K be subgroups of a group G. Exercises 50 and 51 ask you to establish necessary and sufficient criteria for G to appear as the internal direct product of H and K.

50. Let H and K be groups and let $G = H \times K$. Recall that both H and K appear as subgroups of G in a natural way. Show that these subgroups H (actually $H \times \{e\}$) and K (actually $\{e\} \times K$) have the following properties.

a. Every element of G is of the form hk for some $h \in H$ and $k \in K$.

b. $hk = kh$ for all $h \in H$ and $k \in K$. **c.** $H \cap K = \{e\}$.

51. Let H and K be subgroups of a group G satisfying the three properties listed in the preceding exercise. Show that for each $g \in G$, the expression $g = hk$ for $h \in H$ and $k \in K$ is unique. Then let each g be renamed (h, k). Show that, under this renaming, G becomes structurally identical (isomorphic) to $H \times K$.

52. Show that a finite abelian group is not cyclic if and only if it contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

53. Prove that if a finite abelian group has order a power of a prime p, then the order of every element in the group is a power of p. Can the hypothesis of commutativity be dropped? Why, or why not?

54. Let G, H, and K be finitely generated abelian groups. Show that if $G \times K$ is isomorphic to $H \times K$, then $G \simeq H$.