Also, each partition of S gives rise to an equivalence relation $\sim$ on S where $a \sim b$ if and only if a and b are in the same cell of the partition.

Proof We must show that the different cells $\bar{a} = \{x \in S \mid x \sim a\}$ for $a \in S$ do give a partition of S, so that every element of S is in some cell and so that if $a \in \bar{b}$, then $\bar{a} = \bar{b}$. Let $a \in S$. Then $a \in \bar{a}$ by the reflexive condition (1), so a is in *at least one* cell.

Suppose now that a were in a cell $\bar{b}$ also. We need to show that $\bar{a} = \bar{b}$ as sets; this will show that a cannot be in more than one cell. There is a standard way to show that two sets are the same:

Show that each set is a subset of the other.

We show that $\bar{a} \subseteq \bar{b}$. Let $x \in \bar{a}$. Then $x \sim a$. But $a \in \bar{b}$, so $a \sim b$. Then, by the transitive condition (3), $x \sim b$, so $x \in \bar{b}$. Thus $\bar{a} \subseteq \bar{b}$. Now we show that $\bar{b} \subseteq \bar{a}$. Let $y \in \bar{b}$. Then $y \sim b$. But $a \in \bar{b}$, so $a \sim b$ and, by symmetry (2), $b \sim a$. Then by transitivity (3), $y \sim a$, so $y \in \bar{a}$. Hence $\bar{b} \subseteq \bar{a}$ also, so $\bar{b} = \bar{a}$ and our proof is complete. ◆

Each cell in the partition arising from an equivalence relation is an **equivalence class**.

EXERCISES 0

In Exercises 1 through 4, describe the set by listing its elements.

1. $\{x \in \mathbb{R} \mid x^2 = 3\}$

2. $\{m \in \mathbb{Z} \mid m^2 = 3\}$

3. $\{m \in \mathbb{Z} \mid mn = 60 \text{ for some } n \in \mathbb{Z}\}$

4. $\{m \in \mathbb{Z} \mid m^2 - m < 115\}$

In Exercises 5 through 10, decide whether the object described is indeed a set (is well defined). Give an alternate description of each set.

5. $\{n \in \mathbb{Z}^+ \mid n \text{ is a large number}\}$

6. $\{n \in \mathbb{Z} \mid n^2 < 0\}$

7. $\{n \in \mathbb{Z} \mid 39 < n^3 < 57\}$

8. $\{x \in \mathbb{Q} \mid x \text{ is almost an integer}\}$

9. $\{x \in \mathbb{Q} \mid x \text{ may be written with denominator greater than } 100\}$

10. $\{x \in \mathbb{Q} \mid x \text{ may be written with positive denominator less than } 4\}$

11. List the elements in $\{a, b, c\} \times \{1, 2, c\}$.

12. Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. For each relation between A and B given as a subset of $A \times B$, decide whether it is a function mapping A into B. If it is a function, decide whether it is one to one and whether it is onto B.

a. $\{(1, 4), (2, 4), (3, 6)\}$

b. $\{(1, 4), (2, 6), (3, 4)\}$

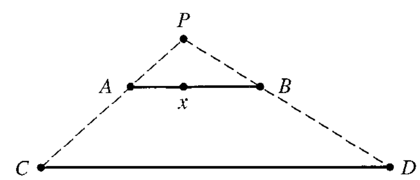
c. $\{(1, 6), (1, 2), (1, 4)\}$

d. $\{(2, 2), (1, 6), (3, 4)\}$

e. $\{(1, 6), (2, 6), (3, 6)\}$

f. $\{(1, 2), (2, 6), (2, 4)\}$

13. Illustrate geometrically that two line segments AB and CD of different length have the same number of points by indicating in Fig. 0.23 what point y of CD might be paired with point x of AB.



0.23 Figure

14. Recall that for $a, b \in \mathbb{R}$ and $a < b$, the **closed interval** $[a, b]$ in $\mathbb{R}$ is defined by $[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}$. Show that the given intervals have the same cardinality by giving a formula for a one-to-one function f mapping the first interval onto the second.

a. $[0, 1]$ and $[0, 2]$ **b.** $[1, 3]$ and $[5, 25]$ **c.** $[a, b]$ and $[c, d]$

15. Show that $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has the same cardinality as $\mathbb{R}$. [*Hint:* Find an elementary function of calculus that maps an interval one to one onto $\mathbb{R}$, and then translate and scale appropriately to make the domain the set S.]

For any set A, we denote by $\mathscr{P}(A)$ the collection of all subsets of A. For example, if $A = \{a, b, c, d\}$, then $\{a, b, d\} \in \mathscr{P}(A)$. The set $\mathscr{P}(A)$ is the **power set** of A. Exercises 16 through 19 deal with the notion of the power set of a set A.

16. List the elements of the power set of the given set and give the cardinality of the power set.

a. $\varnothing$ **b.** $\{a\}$ **c.** $\{a, b\}$ **d.** $\{a, b, c\}$

17. Let A be a finite set, and let $|A| = s$. Based on the preceding exercise, make a conjecture about the value of $|\mathscr{P}(A)|$. Then try to prove your conjecture.

18. For any set A, finite or infinite, let B^A be the set of all functions mapping A into the set $B = \{0, 1\}$. Show that the cardinality of B^A is the same as the cardinality of the set $\mathscr{P}(A)$. [*Hint:* Each element of B^A determines a subset of A in a natural way.]

19. Show that the power set of a set A, finite or infinite, has too many elements to be able to be put in a one-to-one correspondence with A. Explain why this intuitively means that there are an infinite number of infinite cardinal numbers. [*Hint:* Imagine a one-to-one function ϕ mapping A into $\mathscr{P}(A)$ to be given. Show that ϕ cannot be onto $\mathscr{P}(A)$ by considering, for each $x \in A$, whether $x \in \phi(x)$ and using this idea to define a subset S of A that is not in the range of ϕ.] Is *the set of everything* a logically acceptable concept? Why or why not?

20. Let $A = \{1, 2\}$ and let $B = \{3, 4, 5\}$.

a. Illustrate, using A and B, why we consider that $2 + 3 = 5$. Use similar reasoning with sets of your own choice to decide what you would consider to be the value of

i. $3 + \aleph_0$, ii. $\aleph_0 + \aleph_0$.

b. Illustrate why we consider that $2 \cdot 3 = 6$ by plotting the points of $A \times B$ in the plane $\mathbb{R} \times \mathbb{R}$. Use similar reasoning with a figure in the text to decide what you would consider to be the value of $\aleph_0 \cdot \aleph_0$.

21. How many numbers in the interval $0 \le x \le 1$ can be expressed in the form .##, where each # is a digit $0, 1, 2, 3, \cdots, 9$? How many are there of the form .#####? Following this idea, and Exercise 15, decide what you would consider to be the value of $10^{\aleph_0}$. How about $12^{\aleph_0}$ and $2^{\aleph_0}$?

22. Continuing the idea in the preceding exercise and using Exercises 18 and 19, use exponential notation to fill in the three blanks to give a list of five cardinal numbers, each of which is greater than the preceding one.

$$\aleph_0, |\mathbb{R}|, \_\_, \_\_, \_\_.$$

In Exercises 23 through 27, find the number of different partitions of a set having the given number of elements.

23. 1 element **24.** 2 elements **25.** 3 elements

26. 4 elements **27.** 5 elements

28. Consider a partition of a set S. The paragraph following Definition 0.18 explained why the relation

$$x \mathcal{R} y \text{ if and only if } x \text{ and } y \text{ are in the same cell}$$

satisfies the symmetric condition for an equivalence relation. Write similar explanations of why the reflexive and transitive properties are also satisifed.

In Exercises 29 through 34, determine whether the given relation is an equivalence relation on the set. Describe the partition arising from each equivalence relation.

29. $n \mathcal{R} m$ in $\mathbb{Z}$ if $nm > 0$ **30.** $x \mathcal{R} y$ in $\mathbb{R}$ if $x \geq y$

31. $x \mathcal{R} y$ in $\mathbb{R}$ if $|x| = |y|$ **32.** $x \mathcal{R} y$ in $\mathbb{R}$ if $|x - y| \leq 3$

33. $n \mathcal{R} m$ in $\mathbb{Z}^+$ if n and m have the same number of digits in the usual base ten notation

34. $n \mathcal{R} m$ in $\mathbb{Z}^+$ if n and m have the same final digit in the usual base ten notation

35. Using set notation of the form $\{\#, \#, \#, \cdots\}$ for an infinite set, write the residue classes modulo n in $\mathbb{Z}^+$ discussed in Example 0.17 for the indicated value of n.

a. $n = 2$ **b.** $n = 3$ **c.** $n = 5$

36. Let $n \in \mathbb{Z}^+$ and let $\sim$ be defined on $\mathbb{Z}$ by $r \sim s$ if and only if $r - s$ is divisible by n, that is, if and only if $r - s = nq$ for some $q \in \mathbb{Z}$.

a. Show that $\sim$ is an equivalence relation on $\mathbb{Z}$. (It is called "congruence modulo n" just as it was for $\mathbb{Z}^+$. See part b.)

b. Show that, when restricted to the subset $\mathbb{Z}^+$ of $\mathbb{Z}$, this $\sim$ is the equivalence relation, *congruence modulo n*, of Example 0.20.

c. The cells of this partition of $\mathbb{Z}$ are *residue classes modulo n* in $\mathbb{Z}$. Repeat Exercise 35 for the residue classes modulo in $\mathbb{Z}$ rather than in $\mathbb{Z}^+$ using the notation $\{\cdots, \#, \#, \#, \cdots\}$ for these infinite sets.

37. Students often misunderstand the concept of a one-to-one function (mapping). I think I know the reason. You see, a mapping $\phi : A \to B$ has a *direction* associated with it, from A to B. It seems reasonable to expect a one-to-one mapping simply to be a mapping that carries one point of A into one point of B, in the direction indicated by the arrow. But of course, *every* mapping of A into B does this, and Definition 0.12 did not say that at all. With this unfortunate situation in mind, make as good a pedagogical case as you can for calling the functions described in Definition 0.12 *two-to-two functions* instead. (Unfortunately, it is almost impossible to get widely used terminology changed.)